On the Changes in the Dimensions of a Steel Wire when Twisted, and on the Pressure of Distortional Waves in Steel.

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In the Proceedings of the Royal Society* there is an account of some experiments which I made to show that wires when twisted lengthen by an amount proportional to the square of the angle of twist, a result expected from an analysis of the strains in a finite pure shear. In those experiments it was necessary to put considerable loads on the wires.

I have now succeeded in measuring the change in the diameter of a wire when twisted, as well as the longitudinal extension, and have found that the change, a contraction, is also proportional to the square of the angle of twist. It has been now found that the change is sensibly the same for large loads and for the smallest load which could be used, when the wire was sufficiently straightened before being twisted, so that apparently the only function of the load is to straighten the wire.

To measure change in the diameter the wire was fastened at the bottom of a long narrow tube, the "wire tube," filled with water. It passed out from the top of the wire tube through a water-tight leather washer. A capillary glass tube rose vertically from an orifice in the side of the tube, into which it was cemented, and the change of the water level in the capillary when the wire was twisted indicated the change in the volume of the wire within the wire tube.

Description of the Apparatus.

The apparatus used for the measurement of the effects is shown in fig. 1, where, for convenience of representation, various parts are put into the plane of the figure, though actually they were in different planes.

An iron bracket B projected from the wall of the laboratory, and a tripod rested on it, on three levelling screws. The tripod carried a conical bearing for the twisting head-piece T. When the axis of this was exactly vertical, the tripod was fixed to the bracket by clamping screws s, s. The twisting head was provided with a circular plate P, with marks at 90° intervals, which could be set against a fixed index *i*. In practice only whole turns were given, so that only one mark was used, except in one experiment described later.

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* Series A, 1909, vol. 82, p. 546.

At the lower end of T, there was a chuck into which the upper end of the wire was inserted, and a tightening screw made a firm grip. The wire in all cases was very nearly 160·5 cm. long. At its lower end it was gripped by a similar chuck attached to a steel cross-piece C, about 29 cm. long, seen endwise in the figure.

Polished steel plates were screwed on to the vertical sides of this cross-piece near its ends. Four horizontal screws, working in brackets projecting from the wall, and with small steel balls at their ends, were screwed up so as just not to touch the steel plates when there was no twist on the wire. But when a twist was put on, the cross-piece moved up against two of the screws, and was thus fixed in position. Below the cross-piece there was a rod to which was attached another rod carrying a platform *p*, and on this weights could be placed. Each weight was in two semicircular halves. Below the platform was a lead weight S, which I call the sinker, with a volume of 1020 c.c. This hung in a can, and near the can was a water cistern, not shown, connected to it by a rubber tube. When the cistern was pulled up water flowed into the can so as just to cover the sinker and lessen the load by 1020 grm. When the cistern was let down the water flowed back into it and the load increased to its full value. This was used to determine Young's modulus and Poisson's ratio.

A table *t* was fixed to the wall independently of the bracket B, to carry the observing microscope, and the observer sat

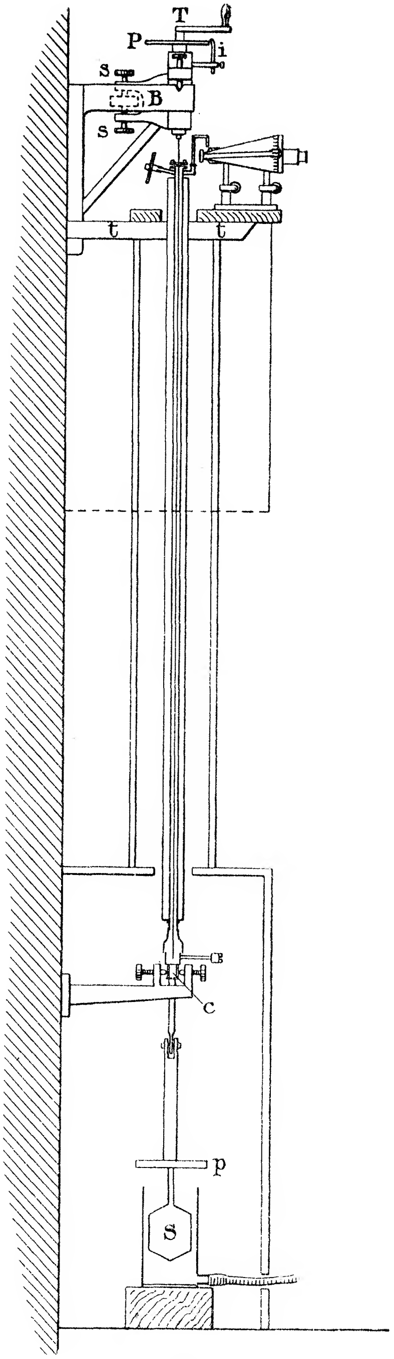

Fig. 1.

on a platform built up about 1·5 metres from the floor. The brass wire tube had an internal diameter about 2 mm. The top of the tube was fixed in a horizontal brass plate, in which was a hole about 0·25 mm. wider than the wire. On this plate rested a well-vaselined leather washer about 1·5 mm. thick, drilled so that it was fairly tight round the wire. On the washer was another brass plate, with a hole in it about 0·35 mm. wider than the wire. Four screws passed freely through holes in this upper plate, and were screwed into the lower plate. Springs between the heads of the screws and the upper surface of the upper plate gave sufficient pressure on the washer. It was found necessary to have the holes in the plates somewhat larger than the wire, in order to adjust the wire and the wire tube both vertical. An arm, not shown in fig. 1, projecting from the lower part of the apparatus, with a sliding weight on it in the plane of the upper side tubes, sufficed to make this vertical adjustment. At first I tried india-rubber washers. They were quite good when first put in, but they deteriorated rather rapidly, and, when they began to perish, they let a small quantity of water out of the tube when the wire moved. The leather washer only required renewal once, when it began to let water escape, and then, on examination, it appeared to be due to action on the wire, which was perceptibly rough on the surface where it emerged from the tube. A short length was cut off the lower end of the wire, and an equal length was let down through the upper chuck, so that once more a smooth part of the wire passed through the washer. There was no further difficulty, and no evidence again of any escape of water.

At the upper end of the wire tube there were two side tubes. A glass capillary tube was cemented into one, and bent as shown on the right in the figure, the vertical branch being about 10 cm. long. When the tube was filled with water, the level in the capillary was adjusted at the level of the microscope about 7 cm. above the level of the washer, as this was about the rise of water in the capillary due to surface tension, and there would therefore be no hydrostatic pressure on the water at the washer level. The tendency to leak would thereby be lessened, but the precaution was probably needless. Into the tube on the left a plunger passed through a leather stuffing box. The plunger had a diameter of 0·2060 cm., and it was driven to or fro by a micrometer screw of $\frac{1}{2}$ mm. pitch. On the head of this screw was a 10 cm. plate, with 500 divisions on its circumference. This plunger was used ordinarily to adjust the level of the water in the capillary. But it was also used to calibrate the capillary. For this calibration, the usual observing microscope was replaced by a microscope cathetometer, and the change of water-level in the capillary was measured for one turn in or one turn out of the screw. Turning always in or always out, it was hardly possible to hit exactly on a whole turn, but a correction could be made, of course, for the fraction of a division in the micrometer head in excess or defect of a whole turn. The mean of 10 measurements when the micrometer was driven inwards 0·5 mm. gave a rise of 11·135 mm., with a range of 0·155 mm. between greatest and least. The mean of 10 measurements when the micrometer was drawn out 0·5 mm. gave a fall of 11·190 mm., with a range of 0·045 mm. The value was taken as 11·16 mm. This gives the cross-section of the capillary as 0·001493 sq. cm., and its diameter as 0·0436 cm.

At the lower end the wire tube was soldered on to a screw cap which could be screwed over the chuck gripping the lower end of the wire.

Below the chuck was a side tube used to fill the wire tube with water. For this purpose the side tube was connected with a flask in which water was boiled. The steam passed up through the crevices in the chuck and out at the plunger tube, from which the plunger was removed. A funnel containing water was connected on to the plunger tube, and when the water in this was boiling freely, through the passage of the steam, the flask was allowed to cool and water was sucked back into the wire tube. When it was full the flask was detached and a cap was screwed on to the lower tube. The plunger was replaced and the capillary, which had been closed meanwhile, was opened. By driving in the plunger the water was raised up to the level of the washer and to any desired point in the capillary.

When the apparatus was not being used the open end of the capillary was under water in a beaker, the plunger being driven in so that the capillary was entirely filled with water. The apparatus thus remained full of water whatever change of temperature might occur. When required for work the beaker was withdrawn and the plunger was screwed out till the meniscus was in the field of view of the microscope.

The wire tube was surrounded by an outer tube about 2·5 cm. diameter, filled with water. This merely served as a means of reducing the effect of outside or inside temperature changes. A wood casing covered with tin foil surrounded the whole from the floor up to the table to lengthen out still further any effects due to temperature change.

To observe the changes of level due to twisting, a microscope with a 1-inch objective and provided with a parallel plate micrometer was used. The micrometer scale was calibrated by means of a millimetre divided to tenths on a standard invar bar. Twelve determinations of 0·4 mm. gave 107·2 micrometer divisions equal to 1 mm., the determinations falling within about 1 per cent. range. Then 1 micrometer division = 0·00933 mm. Since the cross-section of the capillary is 0·001493 sq. cm., one division of the micrometer signifies a change of volume of the water in the wire tube of $1·393 \times 10^{-6}$ c.c.

When it was desired to read the height of the water in the tube the micrometer plate was moved till the cross-wire in the microscope just touched the image of the lowest point of the capillary meniscus. The field was well illuminated by a small lamp behind the capillary, but the image was not always very distinct, and settings of the micrometer could not be trusted to, I think, two or three tenths of a division in some cases, though usually they were more exact. Close to the tube and between it and the micrometer plate a small vertical plate of glass was fixed to the tube at 45° to the line of sight, and this reflected the point of a needle which was also fixed to the tube, so that its image was in the same plane as, and close to, the image of the meniscus. This enabled the observer to note the position of either the meniscus or the needle point without moving the microscope.

When the wire lengthened the wire tube was let down by an equal amount, and the needle point fell. Let us call this fall NP.

At the same time the wire contracted laterally, and the meniscus fell in the tube, and the fall relative to the tube gave the change in volume. The fall observed was that relative to the tube plus that of the tube or NP.

Hence, if the fall observed in the microscope is T the fall relative to the tube is T−NP.

The Wires and their Preparation.

Two piano-steel wires were used in the experiments here described, No. I with a mean diameter 0·0986 cm., the diameters in two planes at right angles being measured with a micrometer every decimetre of its length. The measurements ranged from 0·980 to 0·989 at different points. No. II had a mean diameter of 0·1210 cm., measured in the same way, with a range at different points from 0·1207 to 0·1212.

It was found necessary to straighten these wires, for, unstraightened, they showed the effect with light loads noticed in the previous paper, an apparent shortening on twisting, due, I think, to coiling. To straighten them they were loaded and an electric current was passed through them. No. I was loaded with 50 kgrm., and received a current of 10 ampères. No. II was loaded with 60 kgrm., and received a current of 16 ampères. In each case the wire drew out slightly and then stopped, acquiring a blue temper without rising to a red heat.

Of course the wires became circularly magnetised, but the magnetisation can hardly have contributed to the results here to be described, as these results are of the same character and order as results obtained with heavily loaded unannealed wires in a number of preliminary experiments made before the experiments took their final shape.

A few experiments were made on a hard drawn copper wire, mean diameter 0·1219 cm. (with a range from 0·1216 to 0·1224 cm.).

The Method of Measuring the Lowering on Twisting.

In making any determination of NP or T the following plan was adopted. Suppose, for instance, that the value of T was to be found for four turns of the wire clockwise as seen from above. The position of the meniscus was read for no twist at a given minute, then my assistant put on four turns clockwise—denoted by C_4—then he gave a signal just before, and again exactly at the next half minute, and I set the cross wire on the meniscus at the half minute. The micrometer was read, and the twist was taken off. At the next half minute the micrometer was set as before. Again C_4 was put on, and so on, usually for 32 observations. The first two or three readings were not taken into account, as initially there was usually some irregularity, due probably to settling down in the bearing. The readings were combined in threes in the usual way to give $T = \frac{1}{2}(a+c)-b$ to eliminate as far as possible any march of the zero reading. With the meniscus there was almost always a march, due chiefly to temperature change, for, of course, the arrangement was a very sensitive thermometer. The mean result of the 32 observations was equivalent to 15 or 16 independent determinations. To determine NP the same course was followed, except that the time was not noted. For, though there was often a march in the zero, it was very much smaller, and the observations were made at sufficiently nearly equal intervals of time without noting exact times. This small march was doubtless partly due to temperature change, but also partly due, I believe, to further settling down of the cone of the twisting head into its bearing through slow squeezing out of the oil. In reconstruction I should try the effect of replacing the conical bearing by a ball bearing.

Before the reading was made it was found to be absolutely necessary to move the head-piece some ten or fifteen times to and fro through a small angle—perhaps diminishing from 20°—on each side of the final position. If this was not done the wire did not sink down or rise up to its final position, probably owing to some small friction in the leather washer. After the alternating motion of the head-piece had been given ten or fifteen times no further alternation made any difference in the reading. It was only after finding the necessity of this that I obtained consistent readings.

The maximum and minimum values of NP in a set of 30 determinations usually differed by about 0·4 division, only once rising to 0·8 division.

The maximum and minimum values of T in a set of 30 differed more, as might be expected; the difference averaging 1·7 divisions, and once rising to nearly 5 divisions. This, however, was in the case of the least load, when the weight was probably insufficient to keep the wire tube quite vertical.

Subsidiary Experiments.

The three following subsidiary experiments were made in order to justify the methods of measuring the changes of dimensions on twisting:—

1. To show that the lengthening on twisting is not due to a change in Young's modulus, Y.

This is satisfactorily proved by the experiments on twisting described below. For, suppose that we have a stress P applied to the end of the wire by a load stretching length l by dl when the wire is not twisted, we have $dl = \mathrm{P}l/\mathrm{Y}$. Now let the wire thus loaded be twisted through, say, four turns and let the lowering through twisting be δ. If this is due to a change in Young's modulus to Y′, $dl+\delta = \mathrm{P}l/\mathrm{Y}'$. Whence $\delta = \mathrm{P}l(\mathrm{Y}'^{-1}-\mathrm{Y}^{-1})$, and δ should be proportional to P, whereas it is found to be very nearly the same for loads varying from 5 to 50 kgrm. (approximately).

It appeared worth while, however, to test the question directly, by finding the extension of wire I for very different loads when 1·02 kgrm. was added, first with the wire untwisted, then with the wire twisted through four turns clockwise. The following results, in micrometer divisions, were obtained, each the mean of a number of measurements:—

Table I.

Load.	No twist. Lowering for 1·02 kgrm.	C_4 twist. Lowering for 1·02 kgrm.
18·5	10·55	10·52
38·5	10·68	10·80
48·5	10·50	10·32
Means	10·58	10·57

Then a twist of four turns produced no measurable change in Young's modulus.

2. To show that the rise and fall of the liquid meniscus were due to, and measured, the change in volume of the wire in the wire tube.

The most satisfactory way of showing this appeared to consist in using the apparatus to measure Poisson's ratio σ. The load was altered by 1·02 kgrm. by alternately immersing the sinker in water and letting the water run out.

The rise and fall of the needle point gave the end extension, and Young's modulus Y could then be calculated; while the rise and fall of the meniscus gave the volume change, and the side contraction and Poisson's ratio σ could be calculated. The rigidity modulus n could be obtained from $n = \frac{1}{2}\mathrm{Y}/(1+\sigma)$.

When the load was increased there was some yield of the supporting bracket. To determine its amount, a needle point was fixed on the bracket close to the upper chuck and sighted by the microscope. A series of loads up to 50 kgrm. showed that the lowering per kilogramme was 0·06 division. An addition of 1·02 kgrm., therefore, lowered the bracket by 0·061 division, and this had to be subtracted from the NP reading when used to find Y. In the lowering of the meniscus T—NP, it obviously did not come into consideration. The observed change of volume given by T—NP had to be corrected by a factor about 160·5/156, since only 156 cm. of wire were within the wire tube and 4·5 cm. were outside. The actual length outside varied from 4·2 to 4·6, and the factor was varied accordingly.

No doubt better values of Y and σ might have been obtained with a larger change of load, but, to test the apparatus, it was important to observe lowerings of the same order as those observed in the twisting. In the following Table II the values of T—NP and NP, due to an addition of 1·02 kgrm., are given in micrometer divisions corrected as above described. Each value is the mean of 30. The range between maximum and minimum in a set averaged 1·7 division for T and 0·57 division for NP:—

Table II.—Elastic Moduli and Poisson's Ratio.

Load in kgrm.	NP.	T—NP.	σ.	10^{-12} Y.	10^{-12} n.
Steel Wire I, Diameter 0·0986 cm.					
18·5	10·65	29·86	0·272	2·11	0·830
28·5	10·60	29·57	0·271	2·12	0·835
38·5	10·74	29·37	0·265	2·09	0·828
48·5	10·44	29·05	0·270	2·16	0·849
Mean values...	—	—	0·270	2·12	0·835
Same Wire with C_4 Twist on it.					
48·5	10·27	28·04	0·265	2·18	0·861
Steel Wire II, Diameter 0·1210 cm.					
48·5	7·05	31·22	0·287	2·12	0·825
Hard-drawn Copper Wire, Diameter 0·1219 cm.					
18·5	11·20	61·52	0·331	1·31	0·493

The values found for a steel wire after annealing, given in the paper already referred to,* were $Y = 2{\cdot}06 \times 10^{12}$, $n = 0{\cdot}809 \times 10^{12}$ (by vibration), whence $\sigma = 0{\cdot}273$.

The values for the steel wires are sufficiently near to each other and to the values previously found to show that the tube readings gave, at any rate, very nearly the true changes in volume.

3. To show that the changes were very nearly isothermal.

The change in temperature of a solid sheared adiabatically through ϵ is—

$$d\theta = -\lambda n \theta \epsilon^2 / 2 \mathrm{J} \mathrm{C}_p \rho,$$

where λn is the decrease in rigidity per degree rise, θ is the absolute temperature, C_p is the specific heat, and ρ is the density.

Let us suppose that a steel wire 156 cm. long and 0·05 cm. radius—nearly wire I—is twisted through one turn. It is sufficient to investigate the effect for one turn, for both the adiabatic temperature change and the twisting effects are proportional to the square of the shear, and therefore in a ratio independent of the shear. Then $\epsilon = 2\pi r/156$, where r is the distance of an element from the axis. The mean change in temperature of such a wire is

$$\int_0^a \frac{2\pi r \, d\theta \, dr}{\pi a^2} = -\frac{\pi^2 a^2 \lambda n \theta}{\mathrm{J} \mathrm{C}_p \rho 156^2},$$

where $a = 0{\cdot}05$.

For steel we may put $n = 10^{12}$, $\lambda = 2 \times 10^{-4}$, $\rho = 7{\cdot}8$, $\mathrm{C}_p = 0{\cdot}112$. Taking θ as 300° A. we find the heat in calories developed by the twist to be about -18×10^{-4} calories. Or on untwisting $+18 \times 10^{-4}$ calories.

If this heat were confined to the steel it would alter its temperature by about 1/600° C. and its linear dimensions by about $1{\cdot}8 \times 10^{-8}$ in 1. The twisting through one turn, as will be seen below, alters the radius by about $3{\cdot}19 \times 10^{-7}$ in 1 and the length by about $1{\cdot}72 \times 10^{-6}$. The effects of an adiabatic change of temperature would, therefore, be appreciable compared with the effects of twisting, especially on the radius. But the wire shares its heat, positive or negative, with the water in the wire tube, and here it may produce a serious effect, if it gets no farther than the wire tube, owing to the considerable coefficient of expansion of the water. Let us suppose that the heat or cold is shared with the water in the wire tube so rapidly that both are at one temperature. The water, having approximately three times the volume of the steel, has about 7/9 the heat capacity of steel plus water. So that the water would receive about 14×10^{-4} calories. If a mass of water at a temperature at which its cubical expansion is α receives H calories its change of volume is Hα, whatever the total volume. In our case the temperature was usually about 12° C., at which α is about 10^{-4}. Then the volume change would be about 14×10^{-8} c.c. and since one micrometer division is about 14×10^{-7} c.c. along the capillary, one turn, through thermal effect alone, would produce a fall on twisting and a rise on untwisting of about 0·1 division. The actual change observed on twisting through one turn was about 0·56 division. If then the heat or cold only slowly spread from the wire, or again if it were rapidly shared with the water in the wire tube but only slowly spread thence, the measurements would be seriously affected. But it is obvious that there must, in reality, be a rapid adjustment of temperature between the wire tube and the outer water jacket, and it was important to find out how rapidly the adjustment progressed. Fortunately the wire was insulated from the tube where it passed through the washer, so that it was easy to pass an electric current through it by connecting the terminals of a battery, one to the bracket, the other to the wire tube. Heating currents of the order of 1 to 2 ampères were thus passed along the wire. The current was put on for 2 seconds, the meniscus rushing up meanwhile fairly uniformly, and the point to which it rose was read on the micrometer. Then, 15 seconds after the cut off, the position of the meniscus was read again and the mean of a number of determinations showed that after 15 seconds only 0·032 of the original rise remained. The original rise varied from 7 to 18 divisions with different currents. If the twisting were made instantaneously and the reading of the fall in the tube were made 15 seconds later, about $0{\cdot}032 \times 0{\cdot}1 \div 0{\cdot}56 = 0{\cdot}006$ of the fall would be due to the cooling on twisting. But this is a very considerable over-estimate. The twisting was usually begun 25 seconds before reading and ended more than 15 seconds before. The effect of temperature change may, I think, be estimated at less than 1/300 of the whole. It was impossible to assign even an approximate value to it and as it proved to be so small it was neglected. The effect would have been reduced altogether beyond consideration if the readings had been taken at intervals of one minute, but this would have introduced errors, probably much worse, through irregularities in the temperature march.

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* *Loc. cit.*, p. 554.

In the experiments on Poisson's ratio the adiabatic change of temperature on adding a load which stretches length l by dl is

$$d\theta = -\alpha Y \theta dl / JC_p \rho l,$$

where αY is the change in Young's modulus per degree. The value of α for steel is about 1/4000. This gives the heat for a stretch of 10 divisions as about -9×10^{-3} calories. With uniform temperature of steel and water in the wire tube the water would have about 7×10^{-3} calories, and its effect would be about 0·5 division. After 15 seconds it would be about 0·016 division. As the change of level observed was about 30 divisions the effect is negligible. The direct effect in lengthening or shortening the wire is easily shown to be very much smaller.

Measurement of the Changes of Dimensions on Twisting.

The method of making the measurements has been described already. In the case of Wire I, NP, or the lengthening of the wire, was observed for various loads for four turns and for two turns clockwise twist, denoted by C_4 and C_2, and for four turns and for two turns counter-clockwise twist, denoted by CC_4 and CC_2, each value being the mean of 30 determinations—once or twice of 40—made as described above. As permanent set came in with five turns, four turns was the maximum twist employed. The mean values of T were also determined for the same four twists and T — NP was corrected for the length of wire outside the wire tube. For one load on Wire I the lowerings for C_3 and CC_3 were also observed.

In all cases the lowering w could be represented very nearly by the parabola

$$L(n+c)^2 = w+b,$$

where n is the number of turns put on and L, c, and b are constants, not, of course, the same for NP and T — NP. The constant c represents the fraction of a turn always on the counter-clockwise side of the point of no twist, about which the lowering is symmetrical. Putting $n = -c$, $b = -w$ is a small shortening, or for a counter-clockwise twist c the wire has a minimum length. The existence of c and b is due to want of homogeneity in the wire. They may be explained by supposing that the wire consists of a core and a sheath twisted against each other in the apparently neutral condition as will be shown in the theory given later. Owing to want of exact centering the image "wobbled" somewhat in the field during twisting, and only returned to the same vertical line after a whole number of turns, so that it was futile to attempt to measure b. But there was fairly conclusive evidence that it had a real existence.

According to the theory given, L is the all-important quantity. The internal strain only shifts the vertex of the parabola without altering its size.

To find the constants of a parabola which should fairly represent the results, it was assumed that the curve went through the point $w = 0, n = 0$, so that $Lc^2 = b$.

The equation then becomes

$$L(n^2+2nc) = w.$$

Let w be the lowering for C_n, and w' that for CC_n, then

$$L(n^2-2nc) = w',$$

whence

$$L = (w+w')/2n^2 \quad \text{and} \quad c = (w-w')/4nL = \tfrac{1}{2}n(w-w')/(w+w').$$

The errors are given by

$$\delta L = \delta(w+w')/2n^2, \quad \text{and} \quad \delta c = \delta(w-w')/4nL,$$

assuming that L is without error in c.

If then we find the value of L, say L_4, from the lowerings at C_4 and CC_4, and the value of L, say L_2, from those at C_2 and CC_2, the value of the former should have four times the weight of the latter and we may take the best value of L as $\frac{1}{5}(4L_4 + L_2)$.

The value of c determined from C_4 and CC_4 should have twice the weight of that determined from C_2 and CC_2, and we may take the best value as $\frac{1}{3}(2c_4 + c_2)$.

In the following tables the results are set out. In Table III the lengthening of the wire I is given in micrometer divisions, and below each lengthening the difference, calculated — observed, is put in italics, the calculated values being those given by the parabolas of which the constants are given in Table IV. Similar tables are given for wire II, and for the hard drawn copper wire, but for a single load only, sufficient to secure good centering. After the experience with various loads with wire I, it appeared unnecessary to vary the load in the other cases. With wire II it was not thought advisable to go beyond three turns, and with the copper wire beyond one turn owing to permanent set, which began to be very considerable beyond those limits. The mode of calculating the best parabola was modified accordingly.

Table III.—Lowering NP for Steel Wire I, diameter 0·0986 cm.

Load.	C_4.	C_2.	O.	CC_2.	CC_4.
kgrm.					
48·5	5·095	1·452	0	0·768	4·233
	+0·06	*−0·03*	*0*	*+0·13*	*−0·13*
38·5	5·353	1·422	0	0·818	4·152
	−0·05	*+0·06*	*0*	*+0·06*	*−0·05*
28·5	5·265	1·500	0	0·858	4·259
	+0·01	*+0·03*	*0*	*+0·04*	*−0·10*
18·5	5·382	1·680	0	0·905	4·091
	+0·14	*−0·13*	*0*	*−0·04*	*+0·06*
4·7	5·407	1·742	0	0·797	3·865
	+0·13	*−0·15*	*0*	*−0·03*	*+0·04*

For load 28·5 C_3 was 3·048, and CC_3 was 2·187, and these were taken into account in calculating the parabola, the differences in each case being *+0·05*.

Table IV.—Constants of Parabolas for Table III.

Load.	L.	*c*.	*b*.
48·5	0·289	0·226	0·015
38·5	0·294	0·256	0·019
28·5	0·295	0·237	0·017
18·5	0·302	0·281	0·024
4·7	0·295	0·345	0·035

There was probably some permanent set given in the last two owing to accidental over-twisting of the wire.

The mean value of L is 0·295, and within errors of observation it is independent of the load. It is hardly likely that this is strictly true.

Table V.—Lowering of the Meniscus T—NP for Wire I corrected for length Outside the Tube.

Load.	C_4.	C_2.	O.	CC_2.	CC_4.
48·5	11·69	3·94	0	0·07	6·65
	+0·33	*−0·17*	*0*	*+0·63*	*−0·78*
38·5	11·63	3·94	0	0·84	6·36
	+0·22	*−0·28*	*0*	*+0·06*	*−0·22*
28·5	11·82	4·00	0	1·13	6·38
	−0·25	*−0·28*	*0*	*−0·23*	*+0·04*
18·5	10·99	2·84	0	1·00	6·17
	+0·36	*+0·37*	*0*	*−0·02*	*−0·02*
4·7	10·17	2·54	0	0·99	8·24
	−0·14	*+0·30*	*0*	*+0·57*	*−0·72*

For load 28·5 C_3 was 7·33 and CC_3 was 2·86. These were used in calculating the parabola, and the differences were respectively *−0·01* and *+0·23*.

Table VI.—Constants of Parabolas for Table V.

Load.	L.	*c*.	*b*.
48·5	0·559	0·69	0·26
38·5	0·569	0·61	0·21
28·5	0·578	0·61	0·21
18·5	0·525	0·54	0·15
4·7	0·548	0·29	0·05

As the errors of observation with the last two loads were about double those for the earlier loads, they are only given half the weight in finding L. The value of L is taken as 0·561.

Table VII.—Lowering NP for Steel Wire II, Diameter 0·1210 cm. Two Independent Sets.

	Load.	C_3.	C_2.	O.	CC_2.	CC_3.
I.	48·5	4·922	2·368	0	1·740	4·000
		+0·05	*−0·06*	*0*	*−0·04*	*+0·05*
II.	48·5	4·983	2·243	0	1·708	4·049
		−0·06	*+0·05*	*0*	*−0·01*	*0·00*

Table VIII.—Constants of Parabolas for Table VII.

	L.	*c*.	*b*.
I	0·501	0·154	0·012
II	0·499	0·147	0·011

The mean value of L is 0·500.

Table IX.—Lowering of the Meniscus T−NP for Wire II, Corrected for length outside the Tube.

	Load.	C_3.	C_2.	O.	CC_2.	CC_3.
I.	48·5	15·16	7·73	0	3·90	9·22
		+0·20	*−0·25*	*0*	*−0·31*	*+0·33*
II.	48·5	15·48	7·61	0	3·71	9·55
		+0·04	*−0·06*	*0*	*−0·07*	*+0·10*

Table X.—Constants of Parabolas for Table IX.

	L.	*c*.	*b*.
I	1·385	0·35	0·17
II	1·398	0·35	0·17

The mean value of L is 1·392.

Copper Wire. Diameter, 0·1219 cm.

I was only able to use C_1 and CC_1 owing to permanent set.

The values of NP were 1·043 and 0·415, and the parabola going through these points and the origin is $0{\cdot}73\,(n+0{\cdot}22)^2 = w+0{\cdot}03$.

The values of T—NP, corrected for 4·2 cm. outside the tube, were 8·007 and 1·433, and the parabola is $4{\cdot}72\,(n+0{\cdot}35)^2 = w+0{\cdot}58$.

The work given in the former paper appears to justify the assumption of the parabolic law for copper.

The End Elongation, Side Contraction, and Volume Increase.

Steel Wire I. Diameter, 0·0986 cm.

If w is the end lowering for one turn from the position of minimum length assumed to be L divisions,

$$\begin{aligned} w &= \mathrm{L} \times \text{length of one micrometer division} \\ &= 0{\cdot}295 \times 933 \times 10^{-6} = 2{\cdot}75 \times 10^{-4}\ \text{cm.} \end{aligned}$$

The length is $l = 160{\cdot}5$. Then

$$w/l = 1{\cdot}71 \times 10^{-6}.$$

If u is the decrease in the radius a for one turn,

$$\begin{aligned} 2\pi aul &= \mathrm{L} \times \text{volume of one micrometer division of capillary,} \\ u &= 0{\cdot}561 \times 1{\cdot}393 \times 10^{-6}/2\pi \times 0{\cdot}0493 \times 160{\cdot}5 \\ &= 1{\cdot}57 \times 10^{-8}\ \text{cm.} \end{aligned}$$

The radius is 0·0493 cm. Then

$$u/a = 3{\cdot}19 \times 10^{-7}.$$

The ratio side contraction/end elongation, namely,

$$u/a \div w/l = 0{\cdot}187.$$

If dv is the volume increase in total volume v,

$$\begin{aligned} dv/v &= (\pi a^2 w - 2\pi alu)/\pi a^2 l \\ &= w/l - 2u/a = 1{\cdot}07 \times 10^{-6}. \end{aligned}$$

All the quantities w/l, u/a, dv/v are proportional to the square of the twist from the point of minimum length. The ratio $u/a \div w/l$ is the same for all twists.

Steel Wire II. Diameter, 0·1210 cm.

Using the values given in the tables, we have for one turn—

$$\begin{aligned} w &= 4{\cdot}66 \times 10^{-4}\ \text{cm.}, & u/a &= 5{\cdot}24 \times 10^{-7}, \\ w/l &= 2{\cdot}90 \times 10^{-6}, & u/a \div w/l &= 0{\cdot}181, \\ u &= 3{\cdot}17 \times 10^{-8}\ \text{cm.}, & dv/v &= 1{\cdot}85 \times 10^{-6}. \end{aligned}$$

Copper Wire. Diameter, 0·1219 cm.

The corresponding quantities given by the single set of observations are not of such weight as those for Wires I and II, but I add them here—

$$w = 6{\cdot}81 \times 10^{-4} \text{ cm.}, \qquad u/a = 1{\cdot}75 \times 10^{-6},$$
$$w/l = 4{\cdot}25 \times 10^{-6}, \qquad u/a \div w/l = 0{\cdot}41,$$
$$u = 10{\cdot}7 \times 10^{-8} \text{ cm.}, \qquad dv/v = 0{\cdot}75 \times 10^{-6}.$$

On comparing the results for Wires I and II we see that side contraction ÷ end elongation is very nearly the same for both. The theory given below makes both w/l and u/a proportional to the square of the radius for wires of the same material undergoing the same twist. But as far as these two wires are concerned they are very nearly proportional to $(\text{radius})^{2{\cdot}5}$. I do not think the discrepancy is to be ascribed to experimental error. Perhaps the theory is inadequate, but I think that it is more probable that slight differences in the material not greatly affecting the ordinary elastic moduli may produce very considerable changes in what we may term the secondary moduli, which, in the theory below, are denoted by p and q.

I should like to have taken observations on several more steel wires with a wider range of diameters, but I am not able to continue the work at present.

Experimental Verification of a Reciprocal Relation.

In the 'Philosophical Magazine' for November, 1911, vol. 22, p. 740, Dr. R. A. Houstoun has expressed the reciprocal relation between the stretching and twisting of a wire (confined within limits of reversibility) in the form

$$\left(\frac{d\theta}{d\text{F}}\right)_{\text{G const.}} = \left(\frac{dw}{d\text{G}}\right)_{\text{F const.}}, \tag{1}$$

where F is the end pull and w the increase in length, G the torque, and θ the twist on the wire (I use letters for length and torque differing from Dr. Houstoun's).

As the apparatus only needed small modification it appeared to be worth while to see how nearly this relation was verified, and Wire II was used for the purpose. Incidentally, the value of the rigidity was obtained by the statical torque method.

When the observations needed are worked out it is found that they are identical, as of course was to be expected, with those needed to verify the relation

$$\left(\frac{d\text{F}}{d\theta}\right)_{w\text{ const.}} = \left(\frac{d\text{G}}{dw}\right)_{\theta\text{ const.}}, \tag{2}$$

which is the more direct expression of the Conservation of Energy in these phenomena.

Taking equation (1) we require to know on the left the extra twist $d\theta$ which must be put upon the wire to keep G the same when a load dF is added. For this purpose the wire was initially loaded with 18·5 kgrm., and the head was turned through a right angle. The bar at the bottom was also turned through a right angle from its usual position. On the cross-bar a mirror was fixed reflecting into a telescope a millimetre scale 156·5 cm. away. The ends of the cross-bar were rounded into arcs of a circle with centre in the axis of the wire and radius 14·70 cm. Horizontal threads passed off these arcs to two very light horizontal spiral springs which stretched very uniformly in proportion to the pull up to 40 or 50 grm. These springs were attached to the bases of two travelling microscopes, of which the horizontal scales merely were used to measure any change in stretch. Initially, the wire was without twist, and the position of the cross-bar on the scale was read. It would have been at least very difficult to determine directly the total stretch of the springs required to keep the cross-bar in position when the head was twisted, so the following plan was adopted :—

A half-turn CC was put on by the head-piece, and the springs were stretched so as to bring the cross-bar to its original position. Then a further turn and a half CC was put on, and the *additional* stretch of each spring needed to keep the cross-bar in position was read. This additional stretch multiplied by 4/3 gave the total stretch of the springs, and thus the pull exerted at each end of the cross-arm needed to maintain two turns twist on the wire. The full stretches thus computed were 12·340 cm. on the left and 12·384 cm. on the right, corresponding to pulls, according to previous calibration, of 49·24 grm. and 46·87 grm., mean 48·06 grm. The torque was therefore

$$G = 48{\cdot}06 \times 981 \times 29{\cdot}4 = 1{\cdot}38 \times 10^6 \text{ cm. dynes.}$$

From this the rigidity is

$$n = 0{\cdot}838 \times 10^{12}.$$

The tube method gave $0{\cdot}825 \times 10^{12}$, and the nearness of the two values appears to show that the springs could be trusted fairly well.

A load $dF = 30$ kgrm. was then added, and the torque for two turns was thereby diminished. The springs therefore contracted, and it was observed that they pulled the cross-bar round through 15·85 mm. on the scale—the mean of five different observations ranging from 15·45 to 16·55, or through an angle 0·00507 radian.

Denoting this angle by $\delta\theta$, and the radius of the cross-bar arm by b, and the decrease of torque by δG

$$\frac{\delta G}{G} = \frac{b\delta\theta}{s}, \qquad (3)$$

where s is the whole mean stretch of the springs for two turns.

But we require the twist $d\theta$, which must be put on the wire from its initial two turns, and in the opposite direction from $\delta\theta$, to restore the torque to G. This is given by

$$\frac{\mathrm{G}}{\mathrm{G}-\delta\mathrm{G}} = \frac{\theta+d\theta}{\theta-\delta\theta}, \quad \text{where } \theta = 4\pi,$$

whence, on substituting for $\delta\mathrm{G}/\mathrm{G}$ from (3), we get

$$d\theta = \left(\frac{b\theta}{s}-1\right)\delta\theta, \quad \text{and} \quad \frac{d\theta}{d\mathrm{F}} = \left(\frac{b\theta}{s}-1\right)\frac{\delta\theta}{d\mathrm{F}}. \tag{4}$$

Taking the right hand of equation (1), we require to know the lowering dw for a change $d\mathrm{G}$ in the torque under constant load. We get the lowering from the equation

$$\mathrm{L}\,(n-c)^2 = w+b,$$

giving

$$dw = 2\,\mathrm{L}\,(n-c)\,dn = \mathrm{L}\,(n-c)\frac{d\theta}{\pi}.$$

Also

$$d\mathrm{G} = \frac{\mathrm{G}\,d\theta}{\theta}.$$

Then

$$\frac{dw}{d\mathrm{G}} = \frac{\mathrm{L}\,(n-c)\,\theta}{\pi\mathrm{G}}. \tag{5}$$

Equating (4) and (5), we ought to find

$$\mathrm{L} = \frac{\pi\mathrm{G}}{n-c}\left(\frac{b\theta}{s}-1\right)\frac{\delta\theta}{\theta\,d\mathrm{F}}. \tag{6}$$

Substituting the known values on the right, viz., $\mathrm{G} = 1{\cdot}38\times10^6$, $n = 2$, $c = 0{\cdot}15$, $b = 14{\cdot}7$, $\theta = 4\pi$, $s = 12{\cdot}37$, $\delta\theta = 0{\cdot}00507$, $d\mathrm{F} = 30\times981000$, we get

$$\mathrm{L} = 4{\cdot}48\times10^{-4}.$$

The observed value of L is given as w on p. 548, viz.,

$$\mathrm{L} = 4{\cdot}66\times10^{-4},$$

showing as close an agreement as could be expected, considering the errors of observation.

Taking the second reciprocal relation (2), to find $\left(\frac{d\mathrm{F}}{d\theta}\right)_{w\ \text{const.}}$ we must twist through $d\theta$ and observe dw, and then calculate what load $d\mathrm{F}$ must be removed to restore the original length.

We have

$$dw = \mathrm{L}\,(n-c)\frac{d\theta}{\pi}, \quad \text{and} \quad d\mathrm{F} = \pi a^2\mathrm{Y}\,dw/l.$$

Then

$$\frac{d\mathrm{F}}{d\theta} = \frac{\mathrm{L}\,(n-c)\,a^2\mathrm{Y}}{l}. \tag{7}$$

To find $\left(\frac{dG}{dw}\right)_{\theta\text{ const.}}$, we put on a load W and observe $\delta\theta$. If δG is the diminution in torque at this point and dG the diminution in torque with the original twist θ;

$$\frac{G-\delta G}{G-dG} = \frac{\theta-\delta\theta}{\theta}.$$

Substituting for $\delta G/G$ from (3), this gives

$$dG = \left(\frac{b\theta}{s}-1\right)\frac{G\delta\theta}{\theta},$$

the change in torque for addition W when θ is constant.

The value of dw for this load is

$$dw = lW/\pi a^2 Y,$$

and

$$\left(\frac{dG}{dw}\right)_{\theta\text{ const.}} = \left(\frac{b\theta}{s}-1\right)\frac{\pi a^2 YG}{lW\theta}\delta\theta. \tag{8}$$

Equating (7) and (8) and putting dF for W, we get

$$L = \frac{\pi G}{n-c}\left(\frac{b\theta}{s}-1\right)\frac{\delta\theta}{\theta dF},$$

the same equation as before.

If we could use a wire without any internal strain when untwisted, c would be zero, and we could calculate L, the lowering for one turn, from observations on the torque and load alone.

A Theory of the Changes of Dimension on Twisting: The Stresses in a Finite Pure Shear.

In the paper already referred to* I showed that in a finite pure shear ϵ such as is represented in fig. 2, in which a cube of section ABCD is sheared into a figure of section ABKL through an angle CBK $= \epsilon$, the thicknesses perpendicular to AB and to the plane of the figure remaining constant, the lines of maximum elongation and contraction are, to the order of ϵ^2, at right angles before the shear, making $\epsilon/4$ with the diagonals of the square, as AE and BG. After the shear they are again at right angles to the order of ϵ^2, and make $\epsilon/4$ with the diagonals on the other side as AF and BH. Since we have elongation in one direction AF, and contraction in a direction BH at right angles, the shear may be maintained by a pressure P along BH and a tension Q along AF as far as forces in the plane of the figure are concerned.

If we go to the first order of ϵ only,

$$P = Q = n\epsilon.$$

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* *Loc. cit.*, p. 546.

If we go to the second order we must put

$$P = n\epsilon + p\epsilon^2,$$

where p is a constant to that order.

If we reverse ϵ, P becomes equal to $-Q$, so that we have

$$-Q = -n\epsilon + p\epsilon^2 \quad \text{or} \quad Q = n\epsilon - p\epsilon^2.$$

We can only assume that there is no pressure or tension perpendicular to the plane of the figure, if we neglect ϵ^2. Going to the second order, we have to allow the possibility of a pressure of that order, which we may put as

$$S = q\epsilon^2,$$

where if q is negative the force is a tension.

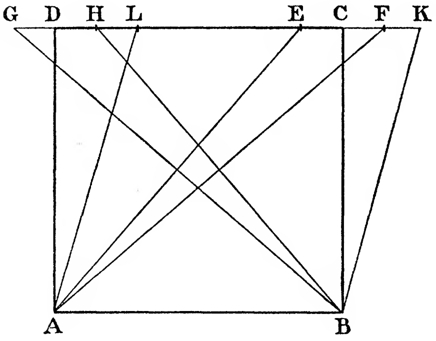

Fig. 2.

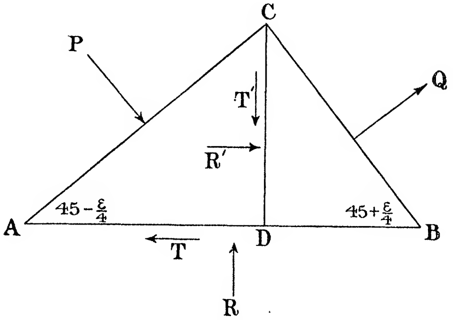

Fig. 3.

Considering the equilibrium of the wedge ABC, fig. 3, with AC in the direction of greatest elongation and BC in that of greatest contraction, I showed that the tangential stress along AB is, to the second order,

$$T = n\epsilon,$$

and that a pressure is required perpendicular to AB given by

$$R = (\tfrac{1}{2}n + p)\,\epsilon^2.$$

The analysis stopped here and was incomplete, as no account was taken of the stresses on the plane CD, fig. 3, perpendicular to AB. It requires to be supplemented as follows:—

Considering the equilibrium of the wedge CDB, let us suppose that on CD there is a tangential stress T′ along CD, and a pressure R′ perpendicular to it.

Resolving all the forces on CDB parallel to DB,

$$R'.\,CB\sin(45+\tfrac{1}{4}\epsilon) - T\,.\,CB\cos(45+\tfrac{1}{4}\epsilon) + Q\,.\,CB\sin(45+\tfrac{1}{4}\epsilon) = 0,$$

whence $$R' = T\cot(45+\tfrac{1}{4}\epsilon) - Q;$$

or, since $T = n\epsilon$ and $\cot(45+\frac{1}{4}\epsilon) = (1-\frac{1}{2}\epsilon)$, neglecting ϵ^2, as it is multiplied by ϵ,

$$R' = n\epsilon(1-\tfrac{1}{2}\epsilon)-n\epsilon+p\epsilon^2 = (-\tfrac{1}{2}n+p)\,\epsilon^2.$$

Resolving parallel to CD,

$$T'.\,CB\sin(45+\tfrac{1}{4}\epsilon)-R\,.\,CB\cos(45+\tfrac{1}{4}\epsilon)-Q\,.\,CB\cos(45+\tfrac{1}{4}\epsilon) = 0,$$

whence
$$T' = (R+Q)\cot(45+\tfrac{1}{4}\epsilon) = \{(\tfrac{1}{2}n+p)\,\epsilon^2+n\epsilon-p\epsilon^2\}\,(1-\tfrac{1}{2}\epsilon)$$
$$= n\epsilon \text{ to the second order.}$$

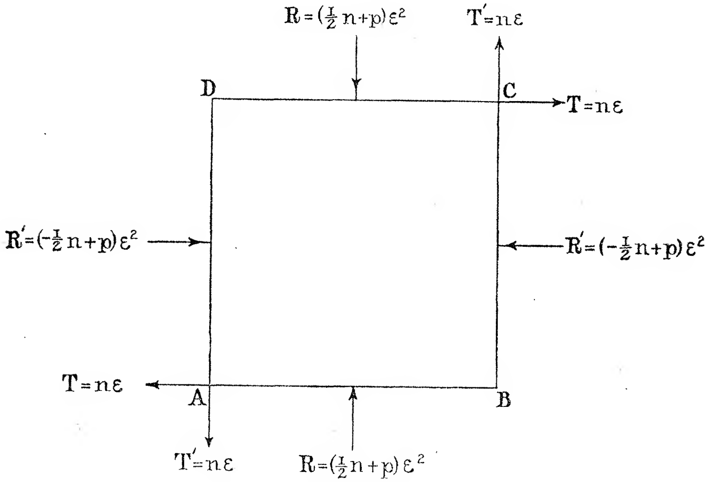
R = ($\frac{1}{2}$n+p) ε²
T′=nε
D
C
T=nε
R′=(-$\frac{1}{2}$n+p) ε²
R′=(-$\frac{1}{2}$n+p) ε²
T=nε
A
B
T′=nε
R=($\frac{1}{2}$n+p) ε²

FIG. 4.

On a unit cube of the material in the sheared condition then, we have, as in fig. 4,

Tangential stresses along AB and CD each.........	$n\epsilon$.
Tangential stresses along AD and BC each.........	$n\epsilon$.
Pressures perpendicular to AB and CD each	$(\frac{1}{2}n+p)\,\epsilon^2$.
Pressures perpendicular to AD and BC each	$(-\frac{1}{2}n+p)\,\epsilon^2$.
And pressures perpendicular to the plane of the figure each $q\epsilon^2$, or, in more convenient form ...	$(q-p)\,\epsilon^2+p\epsilon^2$.

The Strains in a Finite Shear Stress consisting of Tangential Stress T, T′ *only.*

If an element is subjected to the system of stresses just investigated, when we put on to it a system of tensions equal and opposite to the second order pressures we have just found, we leave only the tangential stresses $T = T' = n\epsilon$. The strains due to these tensions must be superposed on the shear ϵ, and we shall then have the strains due to the tangential stresses only.

We have then to examine the strains due to tensions—

$(\frac{1}{2}n+p)\epsilon^2$ on AB and CD (fig. 4).
$(-\frac{1}{2}n+p)\epsilon^2$ on AD and BC.
$(q-p)\epsilon^2+p\epsilon^2$ perpendicular to the plane of the figure.

Through the tension $p\epsilon^2$ on every face we get an extension in all directions $p\epsilon^2/3\mathrm{K}$, where K is the bulk modulus.

The tensions $\frac{1}{2}n\epsilon^2$ on AB and CD and the pressures $\frac{1}{2}n\epsilon^2$ on AD and BC constitute a shear stress giving elongation parallel to BC $\frac{1}{4}n\epsilon^2/n = \frac{1}{4}\epsilon^2$, and a contraction parallel to AB also $\frac{1}{4}\epsilon^2$.

The tensions $(q-p)\epsilon^2$ perpendicular to the plane of the figure give an elongation perpendicular to that plane $\frac{1}{\mathrm{Y}}(q-p)\epsilon^2$, and contractions at right angles, viz., along AB and AD, $\frac{\sigma}{\mathrm{Y}}(q-p)\epsilon^2$, where Y is Young's modulus and σ is Poisson's ratio.

Collecting the results, we have secondary strains accompanying the shear ϵ as follows:—

An elongation parallel to BC $=\frac{1}{4}\epsilon^2+\frac{p\epsilon^2}{3\mathrm{K}}-\frac{\sigma}{\mathrm{Y}}(q-p)\epsilon^2.$

" " AB $=-\frac{1}{4}\epsilon^2+\frac{p\epsilon^2}{3\mathrm{K}}-\frac{\sigma}{\mathrm{Y}}(q-p)\epsilon^2.$

" perpendicular to the plane $=\frac{p\epsilon^2}{3\mathrm{K}}+\frac{1}{\mathrm{Y}}(q-p)\epsilon^2.$

As in the experiments described above Y and σ were determined directly, it will be convenient to replace K from the equation $\frac{1}{3\mathrm{K}}=\frac{1-2\sigma}{\mathrm{Y}}$, and the secondary strains become

$$\left(\frac{1}{4}+\frac{1-\sigma}{\mathrm{Y}}p-\frac{\sigma}{\mathrm{Y}}q\right)\epsilon^2,\quad \left(-\frac{1}{4}+\frac{1-\sigma}{\mathrm{Y}}p-\frac{\sigma}{\mathrm{Y}}q\right)\epsilon^2,\quad \left(-\frac{2\sigma}{\mathrm{Y}}p+\frac{1}{\mathrm{Y}}q\right)\epsilon^2.$$

Equations Representing the Changes in the Dimensions of a Wire Subject to a Torque.

I am indebted to Sir Joseph Larmor for his kindness in indicating how the following equations should be formed and solved. Let us assume that we put on to a wire of length l and radius a a pure shear stress proportional to the distance r from the axis, and twisting length l through angle θ. Then in addition to the shear $\epsilon = r\theta/l$, this stress would produce in an element unconstrained by neighbouring material what we may term "free strains" with the values just found, which we may write as αr^2 radial; βr^2 transverse to the radius; and γr^2 longitudinal; where

$$\left.\begin{aligned} \alpha &= \left(-\frac{2\sigma}{Y}p+\frac{1}{Y}q\right)\frac{\theta^2}{l^2} \\ \beta &= \left(-\frac{1}{4}+\frac{1-\sigma}{Y}p-\frac{\sigma}{Y}q\right)\frac{\theta^2}{l^2} \\ \gamma &= \left(+\frac{1}{4}+\frac{1-\sigma}{Y}p-\frac{\sigma}{Y}q\right)\frac{\theta^2}{l^2} \end{aligned}\right\} \tag{1}$$

If u is the actual radial displacement, and if w is the actual longitudinal displacement, the strains in addition to the shear ϵ are, in cylindrical co-ordinates,

$$du/dr, \qquad u/r, \quad \text{and} \quad dw/dz.$$

The differences between these actual strains and the "free strains," viz.,

$$e = \frac{du}{dr}-\alpha r^2, \qquad f = \frac{u}{r}-\beta r^2, \qquad g = \frac{dw}{dz}-\gamma r^2, \tag{2}$$

imply "secondary stresses" in the wire due to adjustment of strain in neighbouring elements. Let these be denoted by R, Θ, W.

To find R, Θ, and W, we treat e, f, g, as if they were strains in an independent system. Putting $\Delta = e+f+g$, the equations are

$$R = \lambda\Delta+2\mu e, \qquad \Theta = \lambda\Delta+2\mu f, \qquad W = \lambda\Delta+2\mu g, \tag{3}$$

where $\lambda = K-\frac{2}{3}n = \dfrac{\sigma Y}{(1+\sigma)(1-2\sigma)}$ and $\mu = n = \dfrac{Y}{2(1+\sigma)}$.

The forces R, Θ, and W must form a system in equilibrium, there being no external forces to balance. Considering the equilibrium of the element ABCD, fig. 5,

$$d(Rr\delta\theta) = \Theta\delta\theta dr, \quad \text{whence} \quad r\frac{dR}{dr}+R = \Theta. \tag{4}$$

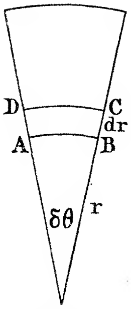

Fig. 5.

We obtain another equation by assuming that the wire is so gripped at each end that sections perpendicular to the axis remain perpendicular to the axis after twisting. Indeed, we have already assumed this in omitting equations for shear stress in (3). Hence w is independent of r and dw/dz is constant over a section for a given wire with a given twist. Let us put

$$dw/dz = h.$$

Further, the load is constant, so that

$$\int_0^a W r\,dr = 0. \tag{5}$$

Substituting in (4) from (3) we obtain

$$r\frac{d^2u}{dr^2}+\frac{du}{dr}-\frac{u}{r}=\frac{2\lambda(\alpha+\beta+\gamma)+6\mu\alpha-2\mu\beta}{\lambda+2\mu}r^2. \tag{6}$$

By putting $u/r = v$, we easily find the solution

$$u = \mathrm{A}r^3+\mathrm{B}r+\mathrm{C}r^{-1}, \tag{7}$$

where
$$\mathrm{A}=\frac{2\lambda(\alpha+\beta+\gamma)+6\mu\alpha-2\mu\beta}{8(\lambda+2\mu)},$$

and B and C are arbitrary constants to be determined by the boundary conditions.

If the wire is unstrained in all parts before twisting, the solution applies with the same constants for all parts.

In order that $u = 0$ when $r = 0$, we must have $\mathrm{C} = 0$, so that

$$u = \mathrm{A}r^3+\mathrm{B}r. \tag{8}$$

When $r = a$, $\mathrm{R} = 0$.

Substituting from (8) in the value of R in (3), and putting $\mathrm{R} = 0$ when $r = a$, we get

$$2(\lambda+\mu)\mathrm{B}+\lambda h = \{\lambda(\alpha+\beta+\gamma)+2\mu\alpha-(4\lambda+6\mu)\mathrm{A}\}\,a^2. \tag{9}$$

From equation (5) we obtain another relation between B and h, when we substitute for u from (8) in W from (3) and integrate from $r = 0$ to $r = a$, viz.,

$$\lambda\mathrm{B}+(\tfrac{1}{2}\lambda+\mu)h = \{\tfrac{1}{4}\lambda(\alpha+\beta+\gamma)+\tfrac{1}{2}\mu\gamma-\lambda\mathrm{A}\}\,a^2, \tag{10}$$

and from (9) and (10) we can find B and h.

Since A is a linear function of α, β, and γ, and each of these is proportional to θ^2, h and B are proportional to θ^2. Substituting for B in (8), u is also proportional to θ^2. The theory, then, gives the parabolic law for the twisting of a wire initially unstrained both for lengthening and for side contraction. It also gives the lengthening and side contraction w/l and u/a for different wires of the same material as proportional to a^2.

So far the theory does not, of course, give any account of the fact that the wires examined are always unsymmetrical, that the effects always date from a point c, on the counter-clockwise side in the wires examined, c being different for w and u. This want of symmetry implies initial internal strain, probably, in reality, very complicated. Let us examine a simple case in which there is a core, radius a, twisted initially against a sheath, outer radius b, and let the opposing twists be respectively θ_c and θ_s. When we put a twist θ from outside on to the core as a whole the core is twisted through $\theta+\theta_c$, and the sheath through $\theta-\theta_s$. For the core and sheath respectively we have

$$u_c = \mathrm{A}r^3+\mathrm{B}r$$
$$u_s = \mathrm{A}'r^3+\mathrm{B}'r+\mathrm{C}'r^{-1},$$

where A is a linear function of α, β, γ, and therefore proportional to $(\theta+\theta_c)^2$ and A′ is the same function of α', β', γ', say, and therefore proportional to $(\theta-\theta_s)^2$.

To find the constants we have

$$\begin{aligned} u_c &= u_s \quad \text{when} \quad r = a, \\ \mathrm{R}_c &= \mathrm{R}_s \quad \text{,,} \quad \text{,,} \\ \mathrm{R}_s &= 0 \quad \text{,,} \quad r = b, \end{aligned}$$

and

$$\int_0^a \mathrm{W}_c r\,dr + \int_a^b \mathrm{W}_s r\,dr = 0.$$

These give us four equations to find B, B′, C′, h of the form (it appears needless to give the detailed work)—

$$\mathrm{B} = \mathrm{P}_1(\theta+\theta_c)^2+\mathrm{Q}_1(\theta-\theta_s)^2, \qquad \mathrm{C}' = \mathrm{P}_3(\theta+\theta_c)^2+\mathrm{Q}_3(\theta-\theta_s)^2,$$

$$\mathrm{B}' = \mathrm{P}_2(\theta+\theta_c)^2+\mathrm{Q}_2(\theta-\theta_s)^2, \qquad h = \mathrm{P}_4(\theta+\theta_c)^2+\mathrm{Q}_4(\theta-\theta_s)^2;$$

and, substituting for A′, B′, C′, in u_s, and putting $r = b$, we get

$$u_b = \mathrm{P}_5(\theta+\theta_c)^2+\mathrm{Q}_5(\theta-\theta_s)^2.$$

Both h and u_b are of the form

$$\mathrm{D}\theta^2+\mathrm{E}\theta+\mathrm{F},$$

where D does not contain θ_c or θ_s. As the parabolas depend only on D, E and F merely giving the position of the vertex, θ_c and θ_s only affect that position.

To find that position we may put $dh/d\theta = 0$ for the one, $du/d\theta = 0$ for the other, and since h and u are different functions of θ_c and θ_s the vertices will be at different points for the two quantities h and u.

Taking this simple case as a guide we shall assume that internal strain only affects the position and not the size of the parabola representing the change of linear dimensions on twisting.

Hence if we could obtain a wire without internal strain we should have

$$\mathrm{L}n^2 = w,$$

where L has the value found in the experiments on the actual, initially strained wire, and we may regard the values u/a and w/l for one turn as the values for a wire initially without internal strain.

The Values of p and q in the Secondary Stresses.

We are now able to find the values of p and q. For the known values of Y and σ enable us to find α, β and γ in equations (1) in terms of p and q

for a known twist, which we shall take as one turn, or as 2π in length $l = 160{\cdot}5$ cm. We also know λ and μ, since

$$\lambda = \sigma \mathrm{Y}/(1+\sigma)(1-2\sigma) \quad \text{and} \quad \mu = n = \mathrm{Y}/2(1+\sigma).$$

Substituting for λ, μ, α, β, and γ we can determine A in terms of p and q. Then from equations (9) and (10) we can find B and h in terms of p and q. Equating $\mathrm{A}a^2+\mathrm{B}$ to the observed value of u/a (which is negative), and h to the observed value of w/l, we have two linear equations in p and q.

The arithmetic is straightforward, though very lengthy, and may be omitted. I have used a slide rule in the calculations.

Using the values of the elastic constants Y and σ from Table II, and the values of u/a and w/l, on p. 548, I find for Wire I

$$p = 1{\cdot}67\times 10^{12}, \qquad q = -0{\cdot}70\times 10^{12},$$

so that the force perpendicular to the plane of the figure in fig. 4 is a tension and not a pressure.

The Pressure in the Direction of Propagation in Distortional Waves and the Longitudinal Waves Produced by the Pressure.

If we had a train of waves purely distortional, that is, a train in which the strain could be represented by a pure shear ϵ, there would be a pressure in the direction of propagation $(\frac{1}{2}n+p)\epsilon^2$. But as ϵ varies from point to point in the train, the pressure due to the shear strain varies, and there must be longitudinal disturbance, longitudinal waves, accompanying the distortional waves. The longitudinal strain implies that the material yields under the pressure, and the pressure will, in general, have a different value from that in a pure shear.

Let us represent the distortional train by

$$\epsilon = \eta \sin \frac{2\pi}{\lambda}(x-vt),$$

where $v^2 = n/\rho$ and η is the amplitude of the shear.

If ξ is the longitudinal displacement at the point where the shear is ϵ, $d\xi/dx$ is the elongation of the element about the point.

Now if we shear a cube, and remove the pressure $(\frac{1}{2}n+p)\epsilon^2$, the cube elongates in that direction, and if the dimensions in the two directions at right angles are maintained the same, the removal of the pressure produces elongation

$$\nu^{-1}(\tfrac{1}{2}n+p)\epsilon^2, \quad \text{where} \quad \nu = \lambda+2\mu = \mathrm{K}+\tfrac{4}{3}n.$$

This we may term the "free elongation" in the direction of propagation on the supposition that there is no change of length at right angles to it.

The pressure due to the shear falls from its full value $(\frac{1}{2}n+p)\epsilon^2$ to 0 while the elongation increases from 0 to its full value $\nu^{-1}(\frac{1}{2}n+p)\epsilon^2$. When the elongation is $d\xi/dx$ the pressure remaining is

$$\mathrm{P} = \frac{\nu^{-1}(\frac{1}{2}n+p)\epsilon^2 - d\xi/dx}{\nu^{-1}(\frac{1}{2}n+p)\epsilon^2}(\tfrac{1}{2}n+p)\epsilon^2 = (\tfrac{1}{2}n+p)\epsilon^2 - \nu\, d\xi/dx$$

$$= (\tfrac{1}{2}n+p)\eta^2\sin^2\frac{2\pi}{\lambda}(x-vt) - \nu\, d\xi/dx.$$

The equation of motion for the longitudinal waves is

$$\rho\frac{d^2\xi}{dt^2} = -\frac{d\mathrm{P}}{dx} = -(\tfrac{1}{2}n+p)\eta^2\frac{2\pi}{\lambda}\sin\frac{4\pi}{\lambda}(x-vt) + \nu\, d^2\xi/dx^2,$$

an equation similar in form to that for the longitudinal waves which I have attempted to show must accompany light waves.*

If we put $$\xi = \mathrm{A}\sin\frac{4\pi}{\lambda}(x-vt-\alpha),$$

and substitute in the above equation, we find on putting $x = 0$, $t = 0$, that $\alpha = 0$, and

$$\mathrm{A} = \frac{(\frac{1}{2}n+p)\eta^2\lambda}{8\pi(\rho v^2-\nu)},$$

or if v' is the velocity of free longitudinal waves, since $\rho v'^2 = \nu$ and $v'^2 > v^2$,

$$\mathrm{A} = -\frac{(\frac{1}{2}n+p)\eta^2\lambda}{8\pi\rho(v'^2-v^2)}.$$

If we substitute for $d\xi/dx$ in P, we get

$$\mathrm{P} = \tfrac{1}{2}(\tfrac{1}{2}n+p)\eta^2\left\{1-\cos\frac{4\pi}{\lambda}(x-vt)\right\} - \nu\mathrm{A}\frac{4\pi}{\lambda}\cos\frac{4\pi}{\lambda}(x-vt).$$

We may regard this as made up by a steady pressure $\frac{1}{2}(\frac{1}{2}n+p)\eta^2$ and a purely periodic pressure, of which the average is zero.

If E is the energy per cubic centimetre at any point in the distortional waves, it is half kinetic energy, half strain energy. The latter is $\frac{1}{2}n\epsilon^2$, so that the total is $n\epsilon^2$ or

$$\mathrm{E} = n\eta^2\sin^2\frac{2\pi}{\lambda}(x-vt) = \tfrac{1}{2}n\eta^2\left\{1-\cos\frac{4\pi}{\lambda}(x-vt)\right\}.$$

Then the average value is

$$\bar{\mathrm{E}} = \tfrac{1}{2}n\eta^2.$$

If we denote the average pressure by $\bar{\mathrm{P}}$,

$$\bar{\mathrm{P}} = \frac{\frac{1}{2}n+p}{n}\bar{\mathrm{E}}.$$

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* 'Roy. Soc. Proc.,' A, vol. 85, p. 474.

If we use the values of n and p found for Wire I, we find

$$\bar{\mathrm{P}} = 2{\cdot}50\,\bar{\mathrm{E}}.$$

If we put the energy per cubic centimetre in the longitudinal waves as $\frac{\rho}{2}\left(\frac{d\xi}{dt}\right)^2 + \frac{\nu}{2}\left(\frac{d\xi}{dx}\right)^2$, we find that

$$\frac{\text{Average energy in longitudinal waves}}{\text{Average energy in distortional waves}} = \frac{1}{8\rho}\,\frac{v'^2+v^2}{(v'^2-v^2)^2}\,\frac{(\frac{1}{2}n+p)^2}{n}\,\eta^2,$$

so that the ratio is proportional to η^2 and therefore in any actual waves it is very small.

The pressures at right angles to the line of propagation will not produce any disturbance in a wave front where η is constant. Round the edges of the wave front, however, where η is diminishing as we go outwards, they may have effects, and it appears likely that they may give rise to disturbances propagated sideways.

I have much pleasure in recording my hearty thanks to Mr. G. O. Harrison, mechanic in the laboratory workshop, for his great help in planning the apparatus used in the experiments described in this paper, for his skill in constructing it, and for his assistance in making the observations.